Adapting Maximum Likelihood Theory to Modern Applications

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CONTENTS.

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The Neglect of Theoretical Statistics</td>
<td>310</td>
</tr>
<tr>
<td>2. The Use of Statistical Methods</td>
<td>311</td>
</tr>
<tr>
<td>3. The Problems of Statistics</td>
<td>313</td>
</tr>
<tr>
<td>4. Criteria of Estimation</td>
<td>316</td>
</tr>
<tr>
<td>5. Examples of the Use of Criterion of Consistency</td>
<td>317</td>
</tr>
<tr>
<td>6. Formal Solution of Problems of Estimation</td>
<td>323</td>
</tr>
<tr>
<td>7. Satisfaction of the Criterion of Sufficiency</td>
<td>330</td>
</tr>
<tr>
<td>8. The Efficiency of the Method of Moments in Fitting Curves of the Pearsonian Type III</td>
<td>332</td>
</tr>
<tr>
<td>9. Location and Scaling of Frequency Curves in general</td>
<td>338</td>
</tr>
<tr>
<td>10. The Efficiency of the Method of Moments in Fitting Pearsonian Curves</td>
<td>342</td>
</tr>
<tr>
<td>11. The Reason for the Efficiency of the Method of Moments in a Small Region surrounding the Normal Curve</td>
<td>355</td>
</tr>
<tr>
<td>12. Discontinuous Distributions</td>
<td>356</td>
</tr>
<tr>
<td>(1) The Poisson Series</td>
<td>359</td>
</tr>
<tr>
<td>(2) Grouped Normal Data</td>
<td>359</td>
</tr>
<tr>
<td>(3) Distribution of Observations in a Dilution Series</td>
<td>363</td>
</tr>
<tr>
<td>13. Summary</td>
<td>366</td>
</tr>
</tbody>
</table>

DEFINITIONS.

Centre of Location.—That abscissa of a frequency curve for which the sampling errors of optimum location are uncorrelated with those of optimum scaling. (9.)

Consistency.—A statistic satisfies the criterion of consistency, if, when it is calculated from the whole population, it is equal to the required parameter. (4.)

Distribution.—Problems of distribution are those in which it is required to calculate the distribution of one, or the simultaneous distribution of a number, of functions of quantities distributed in a known manner. (3.1)
Broad Impact of Classical MLE

Classical MLE: $\hat{\theta}_n = \arg\max_{\theta} \sum_{i=1}^{n} \log p_\theta(X_i)$

• Generality
  - Write down the probability model
  - Maximize the likelihood w.r.t parameter

• Optimality [Fisher, Cramér, Rao, Stein, Hájek, Le Cam, Bickel...]

Online Learning

- We need FAST algorithms to give real-time update and predictions.
- Online advertising, Online recommendation system, Google maps...
Privacy Concerns

• Many data of interest contains sensitive/personal information
  • health records, genetic data, internet browsing history, etc.
Data Privacy in Practice

- Next word prediction (auto-completion)
- Ranking emojis
Towards a general recipe to generate optimal procedures for modern applications.
A First Question

What criterion should we use to define “optimality”? 
Minimax Criterion

A principled way to define “optimality”—minimax criterion [Von Neumann 28, Wald 39]

\[
\min_{\hat{\theta}_n} \max_{\theta \in \Theta} \mathbb{E}_{\theta} \left[ L(\hat{\theta}_n(X_1, \ldots, X_n), \theta) \right]
\]

- $\Theta$: parameter space
- $L$: loss function

Criticism:
- Conservative when $\Theta$ is large.
- Statistician: I do not care worst case. I only care my problem at hand.
“Minimax optimal” for all of machine learning

- Goal: Predict binary label $Y$ from $X$.
- Worst case: $X$ independent of $Y$.
- Random guess is minimax optimal!
What optimality criterion characterizes MLE?

- Classical MLE is not just simply optimal for the worst-case problem.
- It is optimal for problems of all difficulties.
Definition: Local Minimax Risk

\[ M_n := \inf_{\hat{\theta}_n} \sup_{\theta \in \Theta} E_{\theta} \left[ L(\hat{\theta}_n, \theta) \right]. \]

\[ M_n^{\text{loc}}(\theta_0) := \sup_{\theta_1 \in \Theta} \inf_{\hat{\theta}_n} \max_{\theta \in \{\theta_0, \theta_1\}} E_{\theta} \left[ L(\hat{\theta}_n, \theta) \right] \]

\[ \approx \inf_{\hat{\theta}_n} \sup_{\|\theta - \theta_0\| \leq C/\sqrt{n}} E_{\theta} \left[ L(\hat{\theta}_n, \theta) \right]. \]

- A localized quantity characterizing the difficulty of estimation for \( \theta = \theta_0 \).

[Stein 56’, Donoho & Liu 87’, 91’, Cai & Low 15’]
Local Minimax Criterion–the Right Criterion

• Classical local minimax theory: [Stein 56', Donoho & Liu 87', 91', Cai & Low 15']

\[ M_{\text{loc}}(\theta_0) \approx \mathbb{E} L \left( \theta_0 + \frac{1}{\sqrt{n}} W, \theta_0 \right) \text{ where } W \sim N(0, I_{\theta_0}^{-1}). \]

• Optimality of MLE:

\[ \sqrt{n} \left( \hat{\theta}_{n}^{\text{mle}} - \theta_0 \right) \xrightarrow{d} N(0, I_{\theta_0}^{-1}). \]
An Overview of the General Strategy

- Classical local minimax theory ⇒ optimality of classical MLE.

Outline:
1. Private local minimax theory ⇒ Private “MLE”
2. Online local minimax theory ⇒ Online “MLE”
Outline

• Privacy
• Online learning
• Future direction
Privacy
An overview of the strategy

MLE

Classical Local Minimax Theory

Private "MLE"

Private Local Minimax Theory

"+" Privacy
Private Data Analysis

\[ X_1, \ldots, X_n \xrightarrow{q(Z \mid X)} Z_1, \ldots, Z_n \xrightarrow{\hat{\theta}_n(Z_1, \ldots, Z_n)} \]

- Valid \( q \in Q \): \( Z_1, \ldots, Z_n \) preserve privacy of \( X_1, \ldots, X_n \).
Local Minimax Framework for Privacy

\[ M_n^{\text{loc}}(\theta_0) := \sup_{\theta_1 \in \Theta} \inf_{\hat{\theta}_n} \max_{\theta \in \{\theta_0, \theta_1\}} \mathbb{E}_\theta \left[ L(\hat{\theta}_n, \theta) \right]. \]

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\[ M_{n}^{\text{loc}}(\theta_0) := \sup_{\theta_1 \in \Theta} \inf_{\hat{\theta}_n} \max_{\theta \in \{\theta_0, \theta_1\}} \mathbb{E}_\theta \left[ L(\hat{\theta}_n, \theta) \right]. \]

\[ M_{n, \text{priv}}^{\text{loc}}(\theta_0) := \sup_{\theta_1 \in \Theta} \inf_{q \in \mathcal{Q}} \inf_{\hat{\theta}_n} \max_{\theta \in \{\theta_0, \theta_1\}} \mathbb{E}_\theta \left[ L(\hat{\theta}_n, \theta) \right]. \]

- Valid \( q \in \mathcal{Q} \): \( Z_1, \ldots, Z_n \) preserve privacy of \( X_1, \ldots, X_n \).
Definition of Privacy

\[ q(Z \mid X) = \prod_{i=1}^{n} q(z_i \mid x_i, z_{1:(i-1)}) \]

- \( \epsilon \)-Differential Privacy [Dwork, McSherry, Nissim, Smith, 06]: for \( x_i, x'_i, z_{1:(i-1)} \),

\[ \frac{q(z_i \mid x_i, z_{1:(i-1)})}{q(z_i \mid x'_i, z_{1:(i-1)})} \leq \exp(\epsilon). \]

- One can’t distinguish \( x_i \) and \( x'_i \) by looking at \( z_i \), conditioning on \( z_{1:(i-1)} \).
Definition of Privacy

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- One can’t distinguish \( x_i \) and \( x'_i \) by looking at \( z_i \), conditioning on \( z_{1:(i-1)} \).

- Average \( \epsilon \)-Privacy [Mironov 17, Duchi & R. 18]: for \( x_i, x'_i, z_{1:(i-1)} \),

\[ \mathbb{E} q(z_i \mid x_i, z_{1:(i-1)}) \left[ \frac{q(z_i \mid x_i, z_{1:(i-1)})}{q(z_i \mid x'_i, z_{1:(i-1)})} \right] \leq \exp(\epsilon). \]
Private Information $I_{\theta_0, \text{priv}} \neq I_{\theta_0}$

Consider a 1-dim $\mathcal{P} = \{P_\theta\}_{\theta \in \mathbb{R}}$, and the log likelihood $\ell_\theta = \log p_\theta$.

**Theorem (Classical local minimax theory: Donoho & Liu 87, 91)**

$$\mathcal{M}_n^{\text{loc}}(\theta_0) \simeq \mathbb{E}[L(\theta_0 + n^{-1/2}W, \theta_0)] \text{ for } W \sim \mathcal{N}(0, I_{\theta_0}^{-1}),$$

where $I_{\theta_0} = \mathbb{E}[|\dot{\ell}_{\theta_0}|^2]$ is the classical Fisher information.
Private Information $I_{\theta_0,\text{priv}} \neq I_{\theta_0}$

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where $I_{\theta_0} = \mathbb{E}[|\dot{\ell}_\theta_0|^2]$ is the classical Fisher information.

**Theorem (Private local minimax theory: Duchi & R. 18)**

$$\mathcal{M}_{n,\text{priv}}^{\text{loc}}(\theta_0) \simeq \mathbb{E}[L(\theta_0 + (n\epsilon^2)^{-1/2}W, \theta_0)] \text{ for } W \sim \mathcal{N}(0, I_{\theta_0,\text{priv}}^{-1}),$$

where $I_{\theta_0,\text{priv}} = (\mathbb{E}[|\dot{\ell}_\theta_0|])^2$ is defined to be the private information.

**Remark:**

- Our private “MLE” achieves the private information lower bound (Later).
- Superefficiency result [Duchi & R. 18'] (not discussed in this talk).
The cost of privacy: Bernoulli estimation

Consider $X_i \overset{\text{iid}}{\sim} \text{Ber}(\theta)$. \((\mathbb{P}(X = 1) = 1 - \mathbb{P}(X = 0) = \theta)\).

Non-private:

Classical Fisher Information:

\[ I_\theta = (\theta(1 - \theta))^{-1}. \]

Classical MLE $\bar{X}_n$:

\[ \sqrt{n} (\bar{X}_n - \theta) \overset{d}{\to} \mathcal{N}(0, \theta(1 - \theta)). \]
The cost of privacy: Bernoulli estimation

Consider $X_i \overset{\text{iid}}{\sim} \text{Ber}(\theta)$. ($\mathbb{P}(X = 1) = 1 - \mathbb{P}(X = 0) = \theta$).

Private:

Private Information:

$$I_{\theta, \text{priv}} = 1$$

Randomized Response [Warner 65']:

$$Z_i = \epsilon^{-1} \cdot \begin{cases} X_i & \text{w.p. } \frac{1+\epsilon/2}{2} \\ 1 - X_i & \text{w.p. } \frac{1-\epsilon/2}{2} \end{cases}$$

Private “MLE” estimator $\bar{Z}_n$:

$$\sqrt{n\epsilon^2} \left( \bar{Z}_n - \theta \right) \overset{d}{\to} \mathcal{N}(0, 1).$$
The cost of privacy: binary logistic regression

Let \((X_i, Y_i)\) i.i.d satisfy the logistic model:

\[
P(Y = 1 \mid X = x) = \frac{1}{1 + \exp(-\theta x)}.
\]

Prediction error loss:

\[
L(\theta, \theta_0) = \mathbb{E}_{(X,Y) \sim P_{\theta_0}} [|P_{\theta}(Y \mid X) - P_{\theta_0}(Y \mid X)|]
\]

Non-private:

Local risk of classical MLE:

\[
\mathcal{M}_n^{\text{loc}}(\theta_0) \asymp \frac{1}{\sqrt{n}} \exp(-|\theta_0|/2)
\]
The cost of privacy: binary logistic regression

Let \((X_i, Y_i)\) i.i.d satisfy the logistic model:

\[
P(Y = 1 \mid X = x) = \frac{1}{1 + \exp(-\theta x)}.
\]

Prediction error loss:

\[
L(\theta, \theta_0) = \mathbb{E}_{(X,Y) \sim P_{\theta_0}} \left[ |P_{\theta}(Y \mid X) - P_{\theta_0}(Y \mid X)| \right]
\]

Private:

Local risk of Private “MLE”:

\[
M_{n, \text{priv}}^{\text{loc}}(\theta_0) \approx \frac{1}{\sqrt{n\epsilon^2}}
\]
**Challenge:** how to construct private “MLE”?

**Question:** For one-dimensional family \( \{P_\theta\}_{\theta \in \mathbb{R}} \), how do we construct \( \hat{\theta}_{n, \text{priv}} \)

\[
\sqrt{n\epsilon^2}(\hat{\theta}_{n, \text{priv}} - \theta_0) \xrightarrow{d} \mathcal{N}(0, I_{\theta_0, \text{priv}}^{-1}) \quad \text{where} \quad I_{\theta_0, \text{priv}} = (\mathbb{E}[|\dot{\ell}_{\theta_0}|])^2.
\]

**Theorem (Private local minimax theory: Duchi & R. 18)**

\[
M_{n, \text{priv}}^{\text{loc}}(\theta_0) \preceq \mathbb{E}[L(\theta_0 + (n\epsilon^2)^{-1/2}W, \theta_0)] \quad \text{for} \quad W \sim \mathcal{N}(0, I_{\theta_0, \text{priv}}^{-1}).
\]

where \( I_{\theta_0, \text{priv}} = (\mathbb{E}[|\dot{\ell}_{\theta_0}|])^2 \) is the private information.
Challenge: how to construct private “MLE”? 

Bernoulli: $X_i \overset{iid}{\sim} \text{Ber}(\theta)$.

Private MLE:

- **Randomized Response** [Warner 65’]:

  $$Z_i = \epsilon^{-1} \cdot \begin{cases} X_i & \text{w.p. } \frac{1+\epsilon/2}{2} \\ 1 - X_i & \text{w.p. } \frac{1-\epsilon/2}{2} \end{cases}$$

- **Estimation procedure:**

  $$\sqrt{n\epsilon^2} \left( \bar{Z}_n - \theta \right) \xrightarrow{d} \mathcal{N}(0, 1).$$
Challenge: how to construct private “MLE”?

General: \( X_i \overset{iid}{\sim} P_\theta. \sqrt{n\epsilon^2} \left( \hat{\theta}_n - \theta_0 \right) \xrightarrow{d} \text{N}(0, I_{\theta_0,\text{priv}}^{-1}) \)

Idea: reduction to Bernoulli case:

\[
\begin{align*}
X & \xrightarrow{g} g(X) & \text{RR} & Z & \xrightarrow{} \hat{\theta}_{n,\text{priv}} \\
& \in \{0, 1\}
\end{align*}
\]
Challenge: how to construct private “MLE”?

General: \( X_i \overset{\text{iid}}{\sim} P_\theta. \) \( \sqrt{n\epsilon^2} \left( \hat{\theta}_n - \theta_0 \right) \xrightarrow{d} \mathcal{N}(0, I_{\theta_0, \text{priv}}^{-1}) \)

Idea: reduction to Bernoulli case:

\[
\begin{array}{c}
X \\
\in \{0, 1\}
\end{array} \xrightarrow{g} \begin{array}{c} g(X) \\
\text{RR}
\end{array} \xrightarrow{} Z \xrightarrow{} \hat{\theta}_{n, \text{priv}}
\]

Estimating Equation:

\[\hat{\theta}_{n, \text{priv}} = \text{invert}_\theta \left\{ \mathbb{E}_\theta[g(X)] = \bar{Z}_n \right\}.\]
Challenge: how to construct private “MLE”?

General: $X_i \overset{iid}{\sim} P_{\theta}$. $\sqrt{n\epsilon^2} \left( \hat{\theta}_n - \theta_0 \right) \overset{d}{\rightarrow} N(0, I_{\theta_0, \text{priv}}^{-1})$

Idea: reduction to Bernoulli case:

Estimating Equation:

$$\hat{\theta}_{n, \text{priv}} = \text{invert}_{\theta} \left\{ \mathbb{E}_{\theta}[g(X)] = \tilde{Z}_n \right\}.$$
Challenge: how to construct private “MLE”?

General: \( X_i \overset{iid}{\sim} P_{\theta}. \) \( \sqrt{n} \epsilon^2 \left( \hat{\theta}_n - \theta_0 \right) \xrightarrow{d} N(0, I_{\theta_0, \text{priv}}^{-1}) \)

Idea: reduction to Bernoulli case:

\[
\begin{align*}
X & \xrightarrow{g} g(X) \xrightarrow{\text{RR}} Z \xrightarrow{\hat{\theta}_{n, \text{priv}}}
\end{align*}
\]

Estimating Equation:

\( \hat{\theta}_{n, \text{priv}} = \text{invert}_{\theta} \{ \mathbb{E}_{\theta}[g(X)] = \bar{Z}_n \} \).

Delta method:

\[
\text{Var}(\hat{\theta}_{n, \text{priv}}) = \left( \frac{d}{d\theta} \mathbb{E}_{\theta}[g(X)] \right)^{-2} \cdot \text{Var}(\bar{Z}_n)
\]

Requirement on \( g(\cdot) \in \{0, 1\} \):

\[
\frac{d}{d\theta} \mathbb{E}_{\theta}[g(X)] \mid_{\theta = \theta_0} = I_{\theta_0, \text{priv}}^{1/2}
\]
Challenge: how to construct private “MLE”?

General: $X_i \overset{iid}{\sim} P_\theta$. $\sqrt{n} \epsilon^2 \left( \hat{\theta}_n - \theta_0 \right) \overset{d}{\to} N(0, I_{\theta_0,\text{priv}}^{-1})$

Idea: reduction to Bernoulli case:

$$X \xrightarrow{g} g(X) \xrightarrow{\text{RR}} Z \xrightarrow{} \hat{\theta}_{n,\text{priv}}$$

Estimating Equation:

$$\hat{\theta}_{n,\text{priv}} = \text{invert}_\theta \{ E_\theta[g(X)] = \bar{Z}_n \}.$$ 

Delta method:

$$\text{Var}(\hat{\theta}_{n,\text{priv}}) = \left( \frac{d}{d\theta} E_\theta[g(X)] \right)^{-2} \cdot \text{Var}(\bar{Z}_n)$$

Requirement on $g(\cdot) \in \{0, 1\}$:

$$\frac{d}{d\theta} E_\theta[g(X)] |_{\theta=\theta_0} = I_{\theta_0,\text{priv}}^{1/2}. $$

Fact

$$\frac{d}{d\theta} E_\theta \left[ 1_{\hat{\ell}_{\theta_0}(X) \geq 0} \right] |_{\theta=\theta_0} = E_{\theta_0} |\ell_{\theta_0}| = I_{\theta_0,\text{priv}}^{1/2}. $$
Algorithm: private MLE

Divide the data into two groups.

- Privatize the first group of data, get an initializer:
  \[ \hat{\theta}_{n,\text{init}} \xrightarrow{P} \theta_0. \]

- Transform the second group of data \( X_1, \ldots, X_n \) into binaries \( B_1, \ldots, B_n \):
  \[ B_i = 1 \left\{ \ell_{\hat{\theta}_{n,\text{init}}}(X_i) \geq 0 \right\} \]

- Privatize \( B_1, \ldots, B_n \) with randomized response:
  \[
  Z_i = \epsilon^{-1} \cdot \begin{cases} 
  W_i & \text{w.p. } \frac{1+\epsilon/2}{2} \\
  1 - W_i & \text{w.p. } \frac{1-\epsilon/2}{2}
  \end{cases}
  \]

- Construct the final estimator \( \hat{\theta}_{n,\text{priv}} \):
  \[
  \hat{\theta}_{n,\text{priv}} = \text{invert}_\theta \left\{ \mathbb{P}_\theta (\ell_{\hat{\theta}_{n,\text{init}}}(X) \geq 0) = \bar{Z}_n \right\}.
  \]
Extension

- Functionals for high dimensional parametric models \( \{P_\theta\}_{\theta \in \mathbb{R}^p} \).
- Model misspecification: true distribution \( P \not\in \{P_\theta\}_{\theta \in \Theta} \).
  - Ex: find the best linear predictor \( \theta \) without assuming a linear model on \( P \).
Extension

- Functionals for high dimensional parametric models \( \{P_\theta\}_{\theta \in \mathbb{R}^p} \).
- Model misspecification: true distribution \( P \not\in \{P_\theta\}_{\theta \in \Theta} \).
  - Ex: find the best linear predictor \( \theta \) without assuming a linear model on \( P \).
- Private local minimax theory \( \Rightarrow \) private information \( \Rightarrow \) private MLE!
Goal: predicting network structure linking the proteins using a real flow cytometry dataset [Hastie, Tibshirani & Friedman 09]

Logistic regression:

\[
\log \frac{\mathbb{P}(Y = 1 \mid X)}{\mathbb{P}(Y = 0 \mid X)} = \theta^T X.
\]

where \(Y\) is the link prediction and \(X\) is the gene expression.

Treat the raw data as population.

Run vanilla logistic regression on the population (raw data) and get \(\theta^*\).

Simulate new data from the population.

Target: estimate \(\theta^* = (\theta_1^*, \theta_2^*, \ldots, \theta_p^*)\).

Compare non-private MLE, private local and private global minimax [Duchi, Jordan & Wainwright 16'] estimators for estimating \(\theta^* = (\theta_1^*, \theta_2^*, \ldots, \theta_p^*)\).
Flow Cytometry Experiment

Figure: Histograms of errors across $T = 1000$ simulation experiments in estimation of the coordinates $\theta_1^*, \theta_2^*, \ldots, \theta_p^*$ (privacy: $\epsilon = 1$)
### Flow Cytometry Experiment

<table>
<thead>
<tr>
<th>$|\hat{\theta}_n - \theta^<em>|_2 / |\theta^</em>|_2$</th>
<th>Non-private</th>
<th>Private-Local</th>
<th>Private-Global</th>
</tr>
</thead>
<tbody>
<tr>
<td>n/p = 5</td>
<td>42.2%</td>
<td>95%</td>
<td>&gt;100%</td>
</tr>
<tr>
<td>n/p = 20</td>
<td>28.1%</td>
<td>64.8%</td>
<td>82.3%</td>
</tr>
<tr>
<td>n/p = 40</td>
<td>19%</td>
<td>42.5%</td>
<td>69.5%</td>
</tr>
<tr>
<td>n/p = 80</td>
<td>14.3%</td>
<td>30.2%</td>
<td>60%</td>
</tr>
<tr>
<td>n/p = 320</td>
<td>6.8%</td>
<td>13.8%</td>
<td>38.6%</td>
</tr>
<tr>
<td>n/p = 1280</td>
<td>3.4%</td>
<td>6.5%</td>
<td>20.2%</td>
</tr>
</tbody>
</table>

**Table:** Relative error of $\hat{\theta}_n$ across $T = 1000$ simulation experiments (privacy: $\epsilon = 1$).
Insights & Follow-ups

- Private local minimax procedure does lead to improvement.
- Privacy learning is challenging in high dimension or $\epsilon$ is low.
- Impacted Apple’s privacy [Bhowmick et al ’18].
Online Learning
Offline vs. Online Algorithm

\[ \text{minimize}_{\theta} \ R(\theta) := \mathbb{E}_P[\ell(\theta; X)] \]

 Offline algorithm:

Example: empirical risk minimization (ERM)

\[ \hat{\theta}_n = \arg\min_{\theta} \frac{1}{n} \sum_{i=1}^{n} \ell(\theta; X_i) \]
Offline vs. Online Algorithm

\[
\text{minimize}_{\theta} \ R(\theta) := \mathbb{E}_P[\ell(\theta; X)]
\]

**Online algorithm:**

\[
\hat{\theta}_{t+1} = \hat{\theta}_t - \alpha_t \nabla_{\theta} \ell(\hat{\theta}_t; X_t)
\]

Example: stochastic gradient descent \((S_t = \{\hat{\theta}_t\})\)
Find *optimal* online algorithm to solve *convex* and *smooth* problem:

\[
\text{minimize } R(\theta) := \mathbb{E}_P[\ell(\theta; X)] \\
\text{subject to } \theta \in \Theta = \{ c_i(\theta) \leq 0 : i = 1, 2, \ldots, m \}.
\]

Ex: Nonnegative least squares, Ridge, Lasso, (Regularized) Portfolio optimization...
Solution

ERM (MLE)

Local Minimax Theory for ERM (Local Minimax Theory for MLE)

“+” Online Requirement

Online ERM

Online Local Minimax Theory
Local Minimax Theory for Online Optimization

- Loss: \( L \left( \sqrt{n} \left( \hat{\theta}_n - \theta_0 \right) \right) \).

- \( \mathcal{M}^\text{loc}_{\infty,\text{on}}(\mathcal{P}_0) \): online local asymptotic minimax risk [Duchi & R. 18’]

- \( \mathcal{M}^\text{loc}_{\infty,\text{off}}(\mathcal{P}_0) \): offline local asymptotic minimax risk [Hájek & Le Cam 70’, 72’, Levit 76’, Bickel, Klassen, Ritov Wellner 93’, Duchi & R. 18’]

**Theorem (Duchi & R. 18)**

Assume regularity conditions on \( L \). Then

\[
\mathcal{M}^\text{loc}_{\infty,\text{on}}(\mathcal{P}_0) = \mathcal{M}^\text{loc}_{\infty,\text{off}}(\mathcal{P}_0) = \mathbb{E}[L(W)] \text{ for } W \sim \mathcal{N}(0, I_{\mathcal{P}_0}^\dagger) \]

**Takehome Message:**

\[
I_{\mathcal{P}_0} = I_{\mathcal{P}_0,\text{on}} = I_{\mathcal{P}_0,\text{off}} = H \Sigma^\dagger H.
\]
The upper bound

How do we construct the optimal online “ERM”?

$$\sqrt{n} \left( \hat{\theta}_{n,\text{on}} - \theta_0 \right) \rightarrow \mathcal{N}(0, I_{P_0}^\dagger).$$
Unconstrained Problem

Population objective (no constraints):

\[
\minimize_{\theta} R(\theta) = \mathbb{E}_{P_0}[\ell(\theta; X)]
\]

Theorem (Polyak & Judisky 92 + Duchi & R. 18)

\textit{SGD averaging is optimal for unconstrained optimization:}

\[
\sqrt{n} \left( \hat{\theta}_{n,\text{on}} - \theta^* \right) \overset{d}{\to} \mathcal{N}(0, I_{P_0}^\dagger).
\]

Stochastic gradient descent (SGD):

\[
\theta_{t+1} = \theta_t - \alpha_t \nabla \ell(\theta_t; X_t)
\]

Keep track of the running average:

\[
\hat{\theta}_{n,\text{on}} = \bar{\theta}_n
\]
What about constrained optimization problems?

\[
\text{minimize} \quad R(\theta) = \mathbb{E}_{P_0}[\ell(\theta; X)]
\]

subject to \( c_i(\theta) \leq 0, \ i = 1, \ldots, m. \)
A Surprise: Projected-SGD fails
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Projected stochastic gradient descent (PSGD):

$$\theta_{t+1} = \Pi_\Theta (\theta_t - \alpha_t \nabla \ell(\theta_t; X_t))$$

Failure: [Duchi & R. 18]

$$I_{P_0} = H \Sigma^\dagger H$$

where

$$\Sigma = \Pi_{\mathcal{T}_0} \text{Cov}_{P_0}(\nabla \ell(\theta_0; X)) \Pi_{\mathcal{T}_0}$$
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Insight: need identify $C_0$

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A Fix: Dual Averaging?

Dual Averaging (DA) [Nesterov 07']:

\[ z_t = -\frac{1}{t} \sum_{k=1}^t \nabla \ell(\theta_k; X_k) \text{ and } \theta_t = \Pi_\Theta (\alpha_t z_t) \]

Insight: averaging stabilizes noise

\[ z_t \approx -\mathbb{E}[\nabla \ell(\theta_0; X)] + \text{noise}_t. \]

Theorem (Duchi & R. 18)

DA identifies the active constraints, i.e.,

\[ \theta_t \in C_0 \text{ eventually.} \]
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Observation [Duchi & R. 18]:

DA does not adapt to curvature.

Failure:

$$I_{P_0} = H \Sigma^\dagger H.$$
New Algorithm: Riemannian dual averaging

High Level Idea: Alternate between (variants of) DA and Riemannian SGD. (see [Duchi & R. 18’] for details of the algorithm)

\[ \sqrt{n} (\hat{\theta}_{n,RDA} - \theta^*) \xrightarrow{d} N(0, I_{P_0}^\dagger). \]
Online information:

\[ I_{P_0} = H \Sigma^\dagger H. \]

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Adapt to ( \Sigma ) (identify constraints)</th>
<th>Adapt to ( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Projected-SGD</td>
<td>( \times )</td>
<td>( \times )</td>
</tr>
<tr>
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<td>( \checkmark )</td>
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Theorem (Duchi & R. 18)

\[
\sqrt{n} \left( \hat{\theta}_{n, \text{RDA}} - \theta^* \right) \overset{d}{\rightarrow} \mathcal{N}(0, I_{P_0}^\dagger).
\]
Conclusion
Takehome Message

• Towards a general recipe to optimal procedures for modern applications